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1993 J. Phys. A: Math. Gen. 26 L793

(http://iopscience.iop.org/0305-4470/26/17/007)

J. Phys. A: Math. Gen. 26 (1993) L793–L796. Printed in the UK

LETTER TO THE EDITOR

On the transport operator with a linearized kernel

Nassar H S Haidar

Department of Mathematics, American University of Beirut, 850 Third Avenue, 18th Floor, New York, NY 10022, USA, and Lebanese NCSR Nuclear Research Center, Lebanon

Received 2 June 1993

Abstract. Based on a discrete-ordinate approach, we show that the parametrized spectral continuum of the neutron transport operator

$$T=\frac{1}{\mu}-\frac{c}{\mu}\int_{-1}^{1}\sum_{s}(\mu'\to\mu)\,\mathrm{d}\mu'$$

is pseudo-continuous. The dispersion relation for its pseudo-continuities depends on both c and the discrete μ.

In [1], the author reported that the eigenvalue continuum of the neutron transport operator could in principle be pointwise 'perforated' or, i.e., be pseudo-continuous. The supporting Fredholm-based theory and the preliminary conclusions derived from it in [1], have been restricted only to a narrow ray $\{-\varepsilon\leqslant\mu\leqslant\varepsilon;\ \alpha=0\}$, representing the medial ($\alpha=0$) discrete ordinate [2] of the eigenfunction.

In general, however, and over any narrow ray, the neutron scattering Kernel $\sum_s$ $(\mu'\to\mu)$ could always be linearized viz

$$\sum_{s}(\mu'\to\mu)=\tfrac{1}{2}[1+b(\alpha)\mu\mu'];\ \mu,\ \mu'\in[\alpha-\varepsilon,\ \alpha+\varepsilon] \tag{1}$$

where $\varepsilon>0$ is a small enough real number, $\alpha\in[-1+\varepsilon,\ 1-\varepsilon]$, $b(\alpha)\in R$ and R is the set of all reals.

Here we generalize this theory [1] that addresses the medial discrete ordinate, to any discrete ordinate $\{\alpha-\varepsilon\leqslant\mu\leqslant\alpha+\varepsilon;\ -1+\varepsilon\leqslant\alpha\leqslant1-\varepsilon\}$ of the eigenfunction. This will be done however in a way leading only to almost bounded singular kernels and not to the analytically continued kernels, applied, e.g., in the entirely different works of Busbridge [3] or Mullikin [4].

Recall the following eigenvalue problem, associated with the separation of variables solution to the azimuthally symmetric one-speed neutron transport equation [1, 2, 5] in plane geometry.

$$Tf=\frac{1}{\mu}f(\mu)-\frac{c}{\mu}\int_{-1}^{1}\sum_{s}(\mu'\to\mu)f(\mu')\,\mathrm{d}\mu'=\frac{1}{v}f. \tag{2}$$

Let us denote in what follows by ν_i's the discrete inverse eigenvalues of the non-selfadjoint singular transport operator

$$T=\frac{1}{\mu}-\frac{c}{\mu}\int_{-1}^{1}\sum_s(\mu'\to\mu)\,\mathrm{d}\mu' \tag{3}$$

and by $f_i(\mu)$ the associated eigenfunctions, which are also eigenfunctions of

$$f(\mu)=\frac{c\nu}{(\nu-\mu)}\int_{-1}^{1}\sum_s(\mu'\to\mu)f(\mu')\,\mathrm{d}\mu'. \tag{4}$$

Moreover, it is well known (see e.g. [1]) that in addition to a discrete part, the spectrum of this operator contains a ν-continuum of inverse eigenvalues, and we denote by $f_\nu(\mu)$ the associated 'eigenfunctions' or spectral amplitudes.

Consider now (1) in (4) to write the equivalent eigenvalue problem

$$f_\alpha(\mu)=\frac{c}{2}\,\nu\int_{\alpha-\varepsilon}^{\alpha+\varepsilon}\frac{[1+b(\alpha)\mu\mu']}{(\nu-\mu)}f_\alpha(\mu')\,\mathrm{d}\mu'+\frac{c}{2}\,\nu\,\frac{A(\nu,\mu)}{(\nu-\mu)} \tag{5}$$

$$\mu,\mu'\in[\alpha-\varepsilon,\alpha+\varepsilon]\qquad\alpha\in[-1+\varepsilon,1-\varepsilon]$$

in which $f_{\alpha,\mu}(\mu)$ or $f_\alpha(\mu)$ is an eigenfunction and

$$A(\alpha,\mu)=2\int_{-1}^{1}\sum_s(\mu'\to\mu)f(\mu')\,\mathrm{d}\mu'-2\int_{\alpha-\varepsilon}^{\alpha+\varepsilon}[1+b(\alpha)\mu\mu']f_\alpha(\mu')\,\mathrm{d}\mu' \tag{6}$$

is a parameter of the non-homogeneous term that happens to depend also on the ν-inverse eigenvalue. For sufficiently small ε however, $A(\alpha,\mu)$ may be replaced by the averaged parameter

$$A_\alpha=2\int_{-1}^{1}\sum_s(\mu'\to\alpha)f(\mu')\,\mathrm{d}\mu'-2\int_{\alpha-\varepsilon}^{\alpha+\varepsilon}[1+b(\alpha)\alpha\mu']f_\alpha(\mu')\,\mathrm{d}\mu' \tag{7}$$

which is only formally independent of μ. It should be pointed out moreover, that this A_α which is quite different from Case's [5] expansion coefficient, turns out to represent a certain anisotropic generalization of the coefficient in a Haidar's theorem [6] on a Stieltjies integral expansion of the singular solution for the isotropic neutron transport equation.

Let us utilize then the substitutions $\mu=\eta+\alpha$ and $\lambda=\nu^{-1}$ in (5) to rewrite it in the Fredholm equation-like form

$$f_\alpha(\eta)=h(\eta,\hat{\lambda})+\lambda\int_{-\varepsilon}^{\varepsilon}k(\eta,\eta',\lambda)f_\alpha(\eta')\,\mathrm{d}\eta' \tag{8}$$

in which

$$h(\eta,\lambda)=\frac{c}{2}\,\frac{A_\alpha}{[1-\lambda(\eta+\alpha)]} \tag{9}$$

and the non-symmetric singular kernel

$$k(\eta,\eta',\lambda)=\frac{c}{2}\,\frac{1+b(\alpha)[\eta\eta'+\alpha(\eta+\eta')+\alpha^2]}{\lambda[1-\lambda(\eta+\alpha)]} \tag{10}$$

may become bounded,

$$\|K\|^2 = \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} k^2(\eta, \eta', \lambda) \, d\eta \, d\eta' < 1 \tag{11}$$

in a Cauchy's principal value, P, sense only for small enough ε.

Theorem. Let ε be sufficiently small so as $A(\alpha, \mu) = A_\alpha$ and $\|K\|^2 < 1$. Then for any discrete ordinate of the spectral amplitude of the T-operator there holds

$$f_\alpha(\mu) = \frac{c}{2} A_\alpha \left[1 + \frac{S_\alpha(\nu, \mu)}{H_\alpha(\nu)} \right] P \frac{\nu}{(\nu - \mu)} \tag{12}$$

$$\mu \in [\alpha - \varepsilon, \alpha + \varepsilon] \qquad \alpha \in [-1 + \varepsilon, 1 - \varepsilon] \qquad \nu \in [-1, 1]$$

where

$$S_\alpha(\nu, \mu) = c\nu \left\{ [1 + b(\alpha)\nu\mu] \tanh^{-1}\left(\frac{\varepsilon}{\nu - \alpha} \right) - \mu \varepsilon b(\alpha) \right\} \tag{13}$$

$$H_\alpha(\nu) = 1 - c\nu \left\{ (2\alpha - 3\nu)\varepsilon b(\alpha) + [1 + b(\alpha)\nu^2] \tanh^{-1}\left(\frac{\varepsilon}{\nu - \alpha} \right) \right\}. \tag{14}$$

Proof. For such η, we can choose a suitable Banach Space B, containing h as an element and can interpret K as an operator mapping into B itself. Equation (8) may then be solved analytically [7] as a linear non-homogeneous second kind Fredholm integral equation in the form

$$f_{\alpha,\lambda}(\eta) = h(\eta, \lambda) + \lambda \int_{-\varepsilon}^{\varepsilon} R(\eta, \tau, \lambda) h(\eta \ \tau) \, d\tau \tag{15}$$

where in the resolvant

$$R(\eta, \tau, \lambda) = D(\eta, \tau, \lambda)/D(\lambda)$$

$$D(\eta, \tau, \lambda) = K(\eta, \tau, \lambda) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} B_n(\eta, \tau, \lambda)\lambda^n$$

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} C_n \lambda^n$$

$$C_n = \int_{-\varepsilon}^{\varepsilon} B_{n-1}(s, s, \lambda) \, ds$$

$$B_n(\eta, \tau, \lambda) = C_n K(\eta, \tau, \lambda) - n \int_{-\varepsilon}^{\varepsilon} K(\eta, s, \lambda) B_{n-1}(s, \tau, \lambda) \, ds.$$

Since $C_0 = 1$ and $B_0(\eta, \tau, \lambda) = K(\eta, \tau, \lambda)$, then on one hand we have

$$B_1(\eta, \tau, \lambda) = K(\eta, \tau, \lambda) \int_{-\varepsilon}^{\varepsilon} B_0(s, s, \lambda) \, ds - \int_{-\varepsilon}^{\varepsilon} K(\eta, s, \lambda) B_0(s, \tau, \lambda) \, ds. \tag{16}$$

Moreover, for sufficiently small ε, it is possible to assume $\tau=0$ in the right-hand side of (16) to establish that $B_1(\eta, \tau, \lambda)\approx 0$. On the other hand,

$$C_1=\int_{-\varepsilon}^{\varepsilon} K(s, s, \lambda)\, ds=\frac{c}{\lambda^2}\left[1+\frac{b(\alpha)}{\lambda^2}\right]\tanh^{-1}\left(\frac{\lambda\varepsilon}{1-\alpha\varepsilon}\right)+2\alpha b(\alpha)\frac{\varepsilon}{\lambda^2}-3cb(\alpha)\frac{\varepsilon}{\lambda^3}.$$

Now since $B_n(\eta, \tau, \lambda)=0$ and $C_n=0\ \forall n>1$, then

$$R(\eta, \tau, \tau)\approx K(\eta, \tau, \lambda)/(1-C_1\lambda) \tag{17}$$

and the accuracy of this approximation in higher, the smaller is ε.

The required results follow by consideration of (17) in (15), analytical integration and back substitution of $\eta=\mu-\alpha$. □

Corollary. If $\Sigma_s(\mu'\to\mu)$ in T is linearized according to (1.1), then the inverse eigenvalue ν-continuum of this operator is pseudo-continuous at the roots of

$$H_\alpha(\nu)=1-c\nu\left\{(2\alpha-3\nu)\varepsilon b(\alpha)+[1+b(\alpha)\nu^2]\tanh^{-1}\left(\frac{\varepsilon}{\nu-\alpha}\right)\right\}=0. \tag{18}$$ □

This corollary, which is the main result of this communication, relates the number and locations of these pseudo-continuities to such decisive factors as the parametrized discrete ordinate $\mu=\alpha$, the corresponding linear anisotropic scattering coefficient $b(\alpha)$ and the neutron yield $c>0$. Note however that the dispersion relation (18) may possibly have real roots only for values of $b(\alpha)$ and/or c that are large enough to be at least of the order of ε^{-1}.

Finally we point out that the effect of the presence of the reported pseudo-continuities over $[-1, 1]$ on possible exclusion of the $\mu=\nu$ singularities in $\|K\|^2$ appears to remain as a posing interesting question.

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